

The one-loop amplitude for six-gluon scattering

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ABSTRACT: We present results for the six-gluon scattering amplitude at one-loop. Since our method is semi-numerical, it yields the result for arbitrary momenta and helicities of the external gluons. We evaluate the colour-ordered sub-amplitudes with gluons, fermions and scalars running in the internal loop. This is more than sufficient to give a complete description of six-gluon scattering at one-loop in QCD. Combination of these results into amplitudes with $\mathcal{N} = 4$ and $\mathcal{N} = 1$ multiplets of supersymmetric Yang-Mills or with a complex scalar in the internal loops allows comparison with analytic results in the literature. The numerical results for most of the helicity combinations with loops of complex scalars are new.

KEYWORDS: NLO Computations, Hadronic Colliders, QCD.

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1. Introduction

Phenomenology at the LHC often involves high multiplicity final states. For example, backgrounds to Higgs searches involve processes such as $PP \rightarrow W^+W^- + 2$ jets and $PP \rightarrow t\bar{t} + b\bar{b}$. Both these examples involve $2 \rightarrow 4$ scatterings. At leading order (LO) such high multiplicity final state amplitudes can be evaluated using either numerical recursive techniques [1–3] or other numerical and/or algebraic techniques [4–8].

However, $\mathcal{O}(\alpha_S)$, next-to-leading order (NLO) corrections to the scattering amplitudes are desirable. Not only do NLO corrections give a first reliable prediction of total rates, they also give a good error estimate on the shapes of distributions. At NLO the current state of the art for hadron colliders are $2 \rightarrow 3$ processes. Thus NLO predictions for $PP \rightarrow 3$ jets [9, 10] (based on virtual corrections of ref. [11–13]) and $PP \rightarrow V + 2$ jets [14] (based on virtual corrections of ref. [15–17]) are known, and codes for $PP \rightarrow t\bar{t} + \text{jet}$ [18, 19] and $PP \rightarrow H + 2$ jets via gluon fusion [20] are under construction. Other processes such as $PP \rightarrow V_1V_2 + \text{jet}$ and $PP \rightarrow V_1V_2V_3$ are now feasible.

By contrast the consideration of $2 \rightarrow 4$ processes is still in its infancy. In electroweak physics the full one-loop electroweak corrections to $e^+e^- \rightarrow 4$ fermions were calculated in Ref. [21, 22]. However the calculation of NLO $2 \rightarrow 4$ QCD scattering cross sections is currently unexplored. Such a calculation involves both the evaluation of the one-loop six-point virtual corrections and the inclusion of the $2 \rightarrow 5$ scattering bremsstrahlung contributions through Monte Carlo integration.

In this paper we consider the virtual corrections to six-gluon scattering which is relevant for a calculation of $PP \rightarrow 4$ jets. By considering the one-loop corrections to $gg \rightarrow gggg$ we select the most complicated QCD six-point processes. If the amplitude is calculated in terms of Feynman diagrams, the number of diagrams is very large and the gauge cancellations between these diagrams is the most severe. These cancellations could be a concern

in a semi-numerical procedure; the six-gluon amplitude therefore provides a stringent test of the method. In this paper we consider neither the bremsstrahlung contributions, nor the one-loop processes involving external quarks, which are needed to obtain results for a physical cross section.

The technique for the analytic calculation of the one-loop corrections to multi-gluon amplitudes, which is relevant for this paper, is the decomposition of the calculation into simpler pieces with internal loops of $\mathcal{N} = 4$ and $\mathcal{N} = 1$ multiplets of super-symmetric Yang-Mills particles and a residue involving only scalar particles in the loops [11, 23, 24]. After recent advances [25–28], all supersymmetric contributions have been computed analytically, however not all of the scalar contributions for six-gluon amplitudes (or higher) are known yet. We present here numerical results for six-gluon contributions. For super-symmetric pieces we provide completely independent cross-checks of analytical results.

Although all one-loop $2 \rightarrow 2$ and almost all of the currently known $2 \rightarrow 3$ amplitudes were calculated using analytic techniques, we believe that semi-numerical or hybrid numerical/analytic techniques offer promise for more rapid progress. This technique was demonstrated recently for the case of the one-loop Higgs Boson plus four partons amplitude [20].

Many methods have been proposed to calculate NLO amplitudes, both semi-numerical [29–40] or numerical [41, 42]. Of these methods only a few have actually been used to evaluate one-loop amplitudes. Only by using the methods in explicit calculations one can be sure that all numerical issues have been addressed properly.

In section 2 we discuss the colour algebra involved with the evaluation of a six-gluon amplitude. The numerical techniques used in this paper are discussed in section 3, while in section 4 the comparison is made with numerous super-symmetric and the few scalar results, which exist in the literature. Finally, our conclusions in section 5 summarize the paper.

2. Six-gluon amplitude at one-loop

At tree-level, amplitudes with n external gluons can be decomposed into colour-ordered sub-amplitudes, multiplied by a trace of n colour matrices, T^a . The traceless, hermitian, $N_c \times N_c$ matrices, T^a , are the generators of the $SU(N_c)$ algebra. Following the usual conventions for this branch of the QCD literature, they are normalized so that $\text{Tr}(T^a T^b) = \delta^{ab}$. Summing over all non-cyclic permutations the full amplitude $\mathcal{A}_n^{\text{tree}}$ is reconstructed from the sub-amplitudes $A_n^{\text{tree}}(\sigma)$ [1, 43],

$$\mathcal{A}_n^{\text{tree}}(\{p_i, \lambda_i, a_i\}) = g^{n-2} \sum_{\sigma \in S_n/Z_n} \text{Tr}(T^{a_{\sigma(1)}} \dots T^{a_{\sigma(n)}}) A_n^{\text{tree}}(p_{\sigma(1)}^{\lambda_{\sigma(1)}}, \dots, p_{\sigma(n)}^{\lambda_{\sigma(n)}}). \quad (2.1)$$

The momentum, helicity (\pm), and colour index of the i -th external gluon are denoted by p_i , λ_i , and a_i respectively. g is the coupling constant, and S_n/Z_n is the set of $(n-1)!$ non-cyclic permutations of $\{1, \dots, n\}$.

The expansion in colour sub-amplitudes is slightly more complicated at one-loop level. Let us consider the case of massless internal particles of spin $J = 0, 1/2, 1$ corresponding to a complex scalar, a Weyl fermion or a gluon. If all internal particles belong to the adjoint

representation of $SU(N_c)$, the colour decomposition for one-loop n -gluon amplitudes is given by [44],

$$\mathcal{A}_n^{[J]}(\{p_i, h_i, a_i\}) = g^n \sum_{c=1}^{\lfloor n/2 \rfloor + 1} \sum_{\sigma \in S_n / S_{n;c}} \text{Gr}_{n;c}(\sigma) A_{n;c}^{[J]}(\sigma), \quad (2.2)$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x and $S_{n;c}$ is the subset of S_n which leaves the double trace structure in $\text{Gr}_{n;c}(1)$ invariant. The leading-colour structure is simply given by,

$$\text{Gr}_{n;1}(1) = N_c \text{Tr}(T^{a_1} \dots T^{a_n}). \quad (2.3)$$

The subleading-colour structures are given by products of colour traces

$$\text{Gr}_{n;c}(1) = \text{Tr}(T^{a_1} \dots T^{a_{c-1}}) \text{Tr}(T^{a_c} \dots T^{a_n}). \quad (2.4)$$

The subleading sub-amplitudes $A_{n;c>1}$ are determined by the leading ones $A_{n;1}^{[1]}$ through the merging relation [45, 44, 23, 46]

$$A_{n;c>1}^{[1]}(1, 2, \dots, c-1; c, c+1, \dots, n) = (-1)^{c-1} \sum_{\sigma \in \text{OP}\{\alpha\}\{\beta\}} A_{n;1}^{[1]}(\sigma_1, \dots, \sigma_n), \quad (2.5)$$

where $\alpha_i \in \{\alpha\} \equiv \{c-1, c-2, \dots, 2, 1\}$, $\beta_i \in \{\beta\} \equiv \{c, c+1, \dots, n-1, n\}$, and $\text{OP}\{\alpha\}\{\beta\}$ is the set of ordered permutations of $\{1, 2, \dots, n\}$ but with the last element n fixed. The ordered permutations are defined as a set of all mergings of α_i with respect to the β_i , such that the cyclic ordering of the α_i within the set $\{\alpha\}$ and of the β_i within the set $\{\beta\}$ is unchanged. In practice, since n is fixed, no further cycling of the set $\{\beta\}$ is required. Thus a complete description can be given in terms of the leading colour sub-amplitudes $A_{n;1}$ alone.

The contribution of a single flavour of Dirac fermion in the fundamental representation, (relevant for quarks in QCD) is

$$\mathcal{A}_n^{\text{Dirac}}(\{p_i, \lambda_i, a_i\}) = g^n \sum_{\sigma \in S_n / Z_n} \text{Tr}(T^{a_{\sigma(1)}} \dots T^{a_{\sigma(n)}}) A_{n;1}^{[1/2]}(p_{\sigma(1)}^{\lambda_{\sigma(1)}}, \dots, p_{\sigma(n)}^{\lambda_{\sigma(n)}}). \quad (2.6)$$

Simple colour arguments [44] allow one to demonstrate that this colour sub-amplitude is the same as the leading colour sub-amplitude for a single Weyl fermion in the adjoint representation defined in eq. (2.2).

Since the subleading colour amplitudes are not independent, we shall henceforth drop them from our discussion. To simplify the notation we shall also drop the subscripts n and c . The amplitude denoted by A will thus refer to leading colour amplitude with six external gluons.

3. Method of calculation

The method we use is purposely kept as simple as possible. Especially in numerical methods this is desirable for both keeping track of numerical accuracy and code transparency.

To generate all the required Feynman diagrams we use Qgraf [47]. The Qgraf output is easily manipulated using Form [48] to write the amplitude in the form

$$A(1, 2, 3, 4, 5, 6) = \sum_{N=2}^6 \sum_{M=0}^N K_{\mu_1 \dots \mu_M}(p_1, \epsilon_1; \dots; p_6, \epsilon_6) I_N^{\mu_1 \dots \mu_M}(p_1, \dots, p_6), \quad (3.1)$$

where the kinematic tensor K depends on the purely four-dimensional external vectors and contains all the particle and process information. The N -point tensor integrals of rank M are defined in D dimensions as

$$I_N^{\mu_1 \dots \mu_M}(p_1, \dots, p_6) = \int \frac{d^D l}{i\pi^{D/2}} \frac{l^{\mu_1} \dots l^{\mu_M}}{d_1 d_2 \dots d_N}, \quad d_i \equiv (l + q_i)^2, \quad q_i \equiv \sum_{j=1}^i p_j, \quad (3.2)$$

and can be evaluated semi-numerically.

For $N \leq 4$ we use the method of [49, 38, 50] which we already developed, tested and used in the calculation of Higgs Boson plus four partons at one-loop [20]. In general, the basis integrals will contain divergences in $\epsilon = (4 - D)/2$ from soft, collinear and ultraviolet divergences and the answer returned by the semi-numerical procedure will be a Laurent series in inverse powers of ϵ .

For the five (six)-point tensor integrals the method we use relies on the completeness (over-completeness) of the basis of external momenta for a generic phase space point. We therefore use a technique for tensor reduction which generalizes the methods of ref. [51, 52]. This technique is valid as long as the basis of external momenta is complete.¹ Assuming we have a complete basis of external momenta we can select a set of 4 momenta $\{p_{k_1}, p_{k_2}, p_{k_3}, p_{k_4}\}$ which form the basis of the four-dimensional space. We can then decompose the loop momentum

$$l^\mu = \sum_{i=1}^4 l \cdot p_{k_i} v_{k_i}^\mu = V^\mu + \frac{1}{2} \sum_{i=1}^4 (d_{k_i} - d_{k_{i-1}}) v_{k_i}^\mu, \quad (3.3)$$

where the v_{k_i} are defined as linear combinations of the basis vectors

$$v_{k_i}^\mu = \sum_{j=1}^4 [G^{-1}]_{ij} p_{k_j}^\mu, \quad G_{ij} = p_{k_i} \cdot p_{k_j}, \quad (3.4)$$

where G is the Gram matrix and

$$V^\mu = -\frac{1}{2} \sum_{i=1}^4 (r_{k_i} - r_{k_{i-1}}) v_{k_i}^\mu, \quad r_k = q_k^2. \quad (3.5)$$

With this relation it is now easy to reduce an N -point function of rank M to a lower rank N -point function and a set of lower rank $(N - 1)$ -point functions

$$I_N^{\mu_1 \dots \mu_M} = I_N^{\mu_1 \dots \mu_{M-1}} V^{\mu_M} + \frac{1}{2} \sum_{i=1}^4 \left(I_{N, k_i}^{\mu_1 \dots \mu_{M-1}} - I_{N, k_{i-1}}^{\mu_1 \dots \mu_{M-1}} \right) v_{k_i}^{\mu_M}, \quad (3.6)$$

¹For exceptional momentum configurations (such as threshold regions or planar event configurations) this is not the case. Exceptional configurations can be treated using a generalization of the expanded relations proposed in refs. [49, 50]. This is beyond the scope of this paper.

where $I_{N,j}$ is a $(N - 1)$ -point integral originating from I_N with propagator d_j removed. More explicitly, choosing without loss of generality the base set $\{p_1, p_2, p_3, p_4\}$, we get

$$\begin{aligned}
 I_N^{\mu_1 \cdots \mu_M} (p_1, p_2, p_3, p_4, p_5, \dots, p_N) &= \\
 &= I_N^{\mu_1 \cdots \mu_{M-1}} (p_1, p_2, p_3, p_4, p_5, \dots, p_N) V^{\mu_M} (p_1, p_2, p_3, p_4) \\
 &+ \frac{1}{2} (I_{N-1}^{\mu_1 \cdots \mu_{M-1}} (p_1 + p_2, p_3, p_4, p_5, \dots, p_N) - I_{N-1}^{\mu_1 \cdots \mu_{M-1}} (p_2, p_3, p_4, p_5, \dots, p_N)) \\
 &\times v_1^{\mu_M} (p_1, p_2, p_3, p_4) \\
 &+ \frac{1}{2} (I_{N-1}^{\mu_1 \cdots \mu_{M-1}} (p_1, p_2 + p_3, p_4, p_5, \dots, p_N) - I_{N-1}^{\mu_1 \cdots \mu_{M-1}} (p_1 + p_2, p_3, p_4, p_5, \dots, p_N)) \\
 &\times v_2^{\mu_M} (p_1, p_2, p_3, p_4) \\
 &+ \frac{1}{2} (I_{N-1}^{\mu_1 \cdots \mu_{M-1}} (p_1, p_2, p_3 + p_4, p_5, \dots, p_N) - I_{N-1}^{\mu_1 \cdots \mu_{M-1}} (p_1, p_2 + p_3, p_4, p_5, \dots, p_N)) \\
 &\times v_3^{\mu_M} (p_1, p_2, p_3, p_4) \\
 &+ \frac{1}{2} (I_{N-1}^{\mu_1 \cdots \mu_{M-1}} (p_1, p_2, p_3, p_4 + p_5, \dots, p_N) - I_{N-1}^{\mu_1 \cdots \mu_{M-1}} (p_1, p_2, p_3 + p_4, p_5, \dots, p_N)) \\
 &\times v_4^{\mu_M} (p_1, p_2, p_3, p_4). \tag{3.7}
 \end{aligned}$$

For example, applying this relation repeatedly to the tensor six-point integrals we will be left with the scalar six-point integral and five-point tensor integrals. The five-point tensor integrals can be reduced using the same technique. Subsequently we can use the method of [49, 38, 50] to further numerically reduce all remaining integrals to the basis of scalar 2-, 3- and 4-point integrals. This procedure turns out to be efficient and straightforward to implement numerically.

4. Comparison with the literature

Since we have directly calculated the loop amplitudes with internal gluons and fermions we can easily obtain the result for QCD with an arbitrary number n_f of flavours of quarks,

$$A^{\text{QCD}} = A^{[1]} + \frac{n_f}{N} A^{[1/2]}. \tag{4.1}$$

However since the analytic calculations in the literature are presented in terms of supersymmetric theories we need to re-organize our results to compare with other authors.

4.1 Supersymmetry

Since we have calculated the amplitudes with massless spin 1, spin 1/2 and spin 0 particles in the internal loop we can combine our results as follows

$$A^{\mathcal{N}=4} = A^{[1]} + 4A^{[1/2]} + 3A^{[0]}, \tag{4.2}$$

$$A^{\mathcal{N}=1} = A^{[1/2]} + A^{[0]}. \tag{4.3}$$

$A^{\mathcal{N}=4}$, so constructed, describes an amplitude where the full supersymmetric $\mathcal{N} = 4$ multiplet runs in the loop, and $A^{\mathcal{N}=1}$ denotes the contribution from an $\mathcal{N} = 1$ super-multiplet running in the loop.

In analytic calculations the intention is to proceed in the opposite direction. Amplitudes with multiplets of supersymmetric Yang-Mills in internal loops have much improved ultra-violet behavior and are four-dimensional cut-constructible. For this reason, all of these supersymmetric amplitudes have been calculated and most have been presented in a form suitable for numerical evaluation. As far as six-gluon amplitudes with scalars in the loop, $A^{[0]}$, are concerned, three of the needed eight independent helicity amplitudes have been published so far. Only in the helicity combinations where all contributions are known can one reconstruct the ingredients needed for QCD amplitudes,

$$A^{[1]} = A^{\mathcal{N}=4} - 4A^{\mathcal{N}=1} + A^{[0]}, \tag{4.4}$$

$$A^{[1/2]} = A^{\mathcal{N}=1} - A^{[0]}. \tag{4.5}$$

4.2 Numerical results

As a preparatory exercise we performed a check of the four- and five-point gluon one-loop amplitudes. We found agreement with the literature [53, 54, 11].

We now turn to the amplitude for six-gluons which is the main result of this paper. Our numerical program allows the evaluation of the one-loop amplitude at an arbitrary phase space point and for arbitrary helicities. For a general phase space point it is useful to re-scale all momenta so that the momenta of the gluons, (and the elements of the Gram matrix), are of $O(1)$ before performing the tensor reduction. Without loss of generality we can assume that this has been done.

To present our numerical results we choose a particular phase space point with the six momenta p_i chosen as follows, (E, p_x, p_y, p_z) ,

$$\begin{aligned} p_1 &= \frac{\mu}{2}(-1, +\sin\theta, +\cos\theta\sin\phi, +\cos\theta\cos\phi), \\ p_2 &= \frac{\mu}{2}(-1, -\sin\theta, -\cos\theta\sin\phi, -\cos\theta\cos\phi), \\ p_3 &= \frac{\mu}{3}(1, 1, 0, 0), \\ p_4 &= \frac{\mu}{7}(1, \cos\beta, \sin\beta, 0), \\ p_5 &= \frac{\mu}{6}(1, \cos\alpha\cos\beta, \cos\alpha\sin\beta, \sin\alpha), \\ p_6 &= -p_1 - p_2 - p_3 - p_4 - p_5, \end{aligned} \tag{4.6}$$

where $\theta = \pi/4, \phi = \pi/6, \alpha = \pi/3, \cos\beta = -7/19$. Note that the energies of p_1 and p_2 are negative and $p_i^2 = 0$. In order to have energies of $O(1)$ we make the choice for the scale $\mu = n = 6$ [GeV]. As usual, μ also denotes the scale which is used to carry the dimensionality of the D -dimensional integrals. The results presented contain no ultraviolet renormalization.

Analytic results require the specification of eight helicity combinations: all other amplitudes can be obtained by the parity operation or cyclic permutations. We choose these eight combinations to be the two finite amplitudes $(++++, -++++)$, the maximal helicity violating amplitudes $(- - + + +, - + - + +, - + + - +)$, and the next-to-maximal helicity violating amplitudes $(- - - + +, - - + - +, - + - + -)$. These eight

Helicity	$1/\epsilon^2$	$1/\epsilon$	1	[Ref]/(Eq.#)
+++++	0	0	0	
+++++	$(-1.034 + i 2.790)10^{-8}$	$(-9.615 + i 3.708)10^{-8}$	$-(0.826 + i 2.514)10^{-7}$	[SN-A]
-++++	0	0	0	
-++++	$(1.568 + i 2.438)10^{-8}$	$(-0.511 + i 1.129)10^{-7}$	$-(3.073 + i 0.1223)10^{-7}$	[SN-A]
--++++	$-161.917 + i 54.826$ $(-0.933 + i 1.513)10^{-8}$	$-489.024 - i 212.415$ $-(7.655 + i 0.440)10^{-8}$	$-435.281 - i 1162.971$ $-(-0.221 + i 1.834)10^{-7}$	[23]/(4.19) [SN-A]
-+ -+++	$-33.024 + i 44.423$ $(-7.542 + i 0.939)10^{-8}$	$-169.358 + i 33.499$ $-(1.157 + i 0.363)10^{-8}$	$-330.119 - i 229.549$ $-(3.474 + i 2.856)10^{-8}$	[23]/(4.19) [SN-A]
-+ + -++	$-0.5720 - i 3.939$ $(-2.279 + i 1.803)10^{-8}$	$6.929 - i 10.302$ $-(1.176 + i 0.399)10^{-7}$	$28.469 - i 5.058$ $(0.054 - i 3.307)10^{-7}$	[23]/(4.19) [SN-A]
----++	$-6.478 - i 10.407$ $(2.686 - i 1.668)10^{-8}$	$6.825 - i 37.620$ $(1.232 + i 0.554)10^{-7}$	$75.857 - i 47.081$ $(0.020 + i 3.334)10^{-7}$	[24]/(6.19) [SN-A]
--+ -++	$14.074 - i 22.908$ $-(1.619 + i 0.943)10^{-8}$	$80.503 - i 23.464$ $-(1.030 + i 8.234)10^{-8}$	$169.047 + i 93.601$ $(1.560 - i 0.801)10^{-8}$	[24]/(6.24) [SN-A]
-+ - -++	$13.454 + i 13.177$ $(1.045 - i 0.113)10^{-9}$	$3.495 + i 58.632$ $(-0.772 + i 1.652)10^{-8}$	$-88.32 + i 103.340$ $(-7.795 + i 7.881)10^{-8}$	[24]/(6.26) [SN-A]

Table 1: $\mathcal{N}=4$ colour-ordered sub-amplitudes evaluated at the specific point, eq. (4.6). The results are given in the 'tHooft-Veltman regularization scheme. [SN-A] means the difference between the semi-numerical result and the analytical one.

amplitudes would not be sufficient for a numerical evaluation, but the numerical approach allows the evaluation of any helicity configuration at will.

In table 1 we give results for a particular colour sub-amplitude $A^{\mathcal{N}=4}(1, 2, 3, 4, 5, 6)$ for the above eight choices of the helicity. An overall factor of $i c_{\Gamma}$ has been removed from all the results in the tables 1, 2, and 3

$$c_{\Gamma} = \frac{(4\pi)^{\epsilon} \Gamma(1 + \epsilon) \Gamma^2(1 - \epsilon)}{16\pi^2 \Gamma(1 - 2\epsilon)}. \tag{4.7}$$

The results for the $\mathcal{N} = 4$ amplitudes depend on the number of helicities of gluons circulating in internal loops. For a recent description of regularization schemes see, for example, ref. [55]. Our results are presented in the 't Hooft-Veltman scheme. The translation to the four-dimensional helicity scheme is immediate

$$A_{\text{FDH}}^{\mathcal{N}=4} = A_{\text{t-HV}}^{\mathcal{N}=4} + \frac{c_{\Gamma}}{3} A_{\text{tree}}. \tag{4.8}$$

Note that analytic results from the literature are quoted in the four-dimensional helicity scheme, which respects supersymmetry. These results have been translated to the 't Hooft-Veltman scheme using eq. (4.8) before insertion in our tables.

In table 2 we give results for the colour sub-amplitudes $A^{\mathcal{N}=1}(1, 2, 3, 4, 5, 6)$ for the same eight helicity choices and where possible compare with analytical results.² Note that because of the relation

$$A^{\mathcal{N}=1}|_{\text{singular}} = \frac{c_{\Gamma}}{\epsilon} A^{\text{tree}}, \tag{4.9}$$

the column giving the single pole can as well be considered as a listing of the results for the colour-ordered sub-amplitudes at tree graph level (stripped only of the overall factor of i).

²In eq. (5.16) of ref. [24] for the degenerate case $m=j-1=2$ one has $\hat{\mathcal{C}}_m = \{j + 1, \dots, n - 1\}$, as can be seen from figure 8 of this same paper. This point has also been made in ref. [56].

Helicity	$1/\epsilon^2$	$1/\epsilon$	1	[Ref]/(Eq.#)
+++++	0	0	0	
+++++	$(-3.470 + i 9.320)10^{-9}$	$(-3.226 + i 1.253)10^{-8}$	$-(3.899 + i 8.969)10^{-8}$	[SN-A]
-++++	0	0	0	
-++++	$(5.228 + i 8.127)10^{-9}$	$(-1.678 + i 3.775)10^{-8}$	$-(1.013 + i 0.2066)10^{-7}$	[SN-A]
-+ + + +	0	$26.986 - i 9.1376$	$101.825 - i 52.222$	[24]/(5.9)
-+ + + +	$(-3.297 + i 5.194)10^{-9}$	$-(-2.104 + i 0.344)10^{-8}$	$(0.949 - i 4.895)10^{-8}$	[SN-A]
-+ - + +	0	$5.504 - i 7.404$	$21.811 - i 29.051$	[24]/(5.12)
-+ - + +	$(-1.847 + i 0.8566)10^{-10}$	$-(6.141 + i 4.633)10^{-10}$	$(3.095 + i 2.138)10^{-7}$	[SN-A]
-+ + - +	0	$0.09533 + i 0.6565$	$-2.183 + i 3.260$	[24]/(5.12)
-+ + - +	$(-7.599 + i 6.018)10^{-9}$	$-(3.929 + i 1.304)10^{-8}$	$(0.008 - i 1.100)10^{-7}$	[SN-A]
- - - + +	0	$1.080 + i 1.735$	$0.722 + i 5.285$	[25]/(9)
- - - + +	$(8.965 - i 5.555)10^{-9}$	$(4.107 + i 1.858)10^{-8}$	$(0.002 + i 1.114)10^{-7}$	[SN-A]
- - + - +	0	$-2.346 + i 3.819$	$-2.238 + i 17.687$	[28]/(5.4,2.3)
- - + - +	$(-5.351 - i 2.825)10^{-9}$	$0.204 + i 2.204)10^{-8}$	$(-3.891 + i 1.237)10^{-8}$	[SN-A]
- + - + -	0	$-2.242 - i 2.196$	$-1.721 - i 7.433$	[28]/(5.13,2.3)
- + - + -	$(1.124 - i 0.2060)10^{-10}$	$0.717 + i 2.947)10^{-9}$	$(-1.049 + i 3.134)10^{-8}$	[SN-A]

Table 2: $\mathcal{N}=1$ colour-ordered sub-amplitudes evaluated at the specific point, eq. (4.6). [SN] means that the result is obtained using our semi-numerical code, while [SN-A] denotes the difference between the semi-numerical result and the analytical one.

Helicity	$1/\epsilon^2$	$1/\epsilon$	1	[Ref]/(Eq.#)
+++++	0	0	$(4.867 + i 2.092)10^{-1}$	[26]/(4.3)
+++++	$(3.672 + i 9.749)10^{-9}$	$(-3.404 + i 1.238)10^{-8}$	$-(3.016 + i 9.169)10^{-8}$	[SN-A]
-++++	0	0	$-3.194 + i 0.6503$	[26]/(4.10)
-++++	$(5.921 + i 8.411)10^{-9}$	$(-1.606 + i 4.051)10^{-8}$	$-(1.086 + i 0.038)10^{-7}$	[SN-A]
-- + + +	0	$8.995 - i 3.046$	$43.089 - i 20.288$	[27]/(4.27,4.28)
-- + + +	$(1.280 + i 0.002)10^{-8}$	$(2.768 + i 4.232)10^{-8}$	$(-1.004 + i 0.955)10^{-7}$	[SN-A]
-+ - + +	$(1.045 - i 0.580)10^{-8}$	$1.835 - i 2.468$	$9.752 - i 11.791$	[SN]
-+ + - +	$(-7.791 + i 6.717)10^{-9}$	$3.178 \cdot 10^{-2} + i 0.2188$	$-1.447 + i 0.1955$	[SN]
- - - + +	$(8.934 - i 5.359)10^{-9}$	$0.3599 + i 0.5782$	$0.5617 + i 5.8166$	[SN]
- - + - +	$(0.1016 + i 1.276)10^{-8}$	$-0.7819 + i 1.273$	$-0.6249 + i 6.552$	[SN]
- + - + -	$(1.065 - i 0.5417)10^{-8}$	$-0.7475 - i 0.7321$	$-1.298 - i 3.255$	[SN]

Table 3: One loop six gluon colour-ordered sub-amplitudes with a scalar loop evaluated the specific point eq. (4.6). [SN] means that the result is obtained using our semi-numerical code, while [SN-A] denotes the difference between the semi-numerical result and the analytical one.

We note that for two of the helicity amplitudes $- - + - + +$ and $- + - + - +$ we were unable to evaluate the analytic results numerically from the published version of ref. [28]. This was due to the fact that calculating the residue of certain poles as required by the formula in ref. [28], resulted in zero value denominators of sub-expressions.³ Subsequent to the appearance of the archive version of our paper, the authors of [28] have submitted a revised version (v4). We are now in agreement with the numbers derived from v4 as shown in table 2.

Lastly in table 3 we give results for the colour sub-amplitudes $A^{[0]}(1, 2, 3, 4, 5, 6)$ for scalar gluons, for the same eight helicity choices.⁴ For all amplitudes for which no analytic result exists, we checked the gauge invariance of the amplitudes by changing the gluon polarization. The gauge invariance was obeyed with a numerical accuracy of $\mathcal{O}(10^{-8})$. To evaluate a single colour-ordered sub-amplitude for a complex scalar took 9 seconds on a

³We thank the authors of ref. [28] for confirming that there are problems with the numerical evaluation of the formula for these amplitudes in the published version of their paper.

⁴In ref. [27] [v1-v3] the definition of F_f has an overall sign missing, a typographical error not present in the original calculation of the $\mathcal{N} = 1$ term in ref. [24].

2.8GHz Pentium processor. To evaluate the complete set of 64 possible helicities will be less than 64 times longer, because the scalar integrals are stored during the calculation of the first amplitude are applicable to all other configurations with the same external momenta.

We have not addressed the issue of the numerical stability of our code in the exceptional regions. This issue has been addressed in detail in ref. [50] for a five-leg process. We believe that the issue can be addressed using similar methods or indeed by simply interpolating over the singular regions.

5. Conclusions

In this paper we have presented numerical results which demonstrate that the complete one-loop amplitude for six-gluon scattering is now known numerically. By forming multiplets of SUSY Yang Mills in the internal loops, we were able to compare with most of the known analytic results. In addition, we have presented numerical results for amplitudes which are currently completely unknown. Note that the analytic and semi-numerical results are complementary. The hardest piece to calculate analytically is the scalar contribution $A^{[0]}$, which is the easiest for the semi-numerical approach. Thus it is possible that a numerical code involving both semi-numerical and analytic results will be the most efficient and expedient. Our results demonstrate the power of the semi-numerical method, which can supplant the analytic method where it is too arduous and provide a completely independent check where analytic results already exist.

After inclusion of the one-loop corrections to the other parton subprocesses involving quarks and treatment of the exceptional configurations, it would be possible to proceed to a NLO evaluation of the rate for four jet production. We intend to use these methods to calculate NLO corrections to other processes which we consider to be of more pressing phenomenological interest.

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